Self-similar parabolic beam generation and propagation

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(Received 10 February 2005; published 12 July 2005)

The (1+1)-dimensional and (2+1)-dimensional amplified nonlinear Schrödinger equations incorporating diffraction, Kerr nonlinearity, and gain are solved analytically and numerically. An asymptotic solution is found corresponding to self-similar propagation of a beam with parabolic amplitude and phase profiles. While the (1+1)-dimensional solution is directly analogous to parabolic pulse propagation in nonlinear dispersive media, the existence of self-similar propagation in (2+1) dimensions is a nontrivial question, given that spatial solitons are unstable in bulk media with nonsaturating nonlinearities. We show that self-similar parabolic beams are possible in such media with gain and a negative nonlinear index.

DOI: 10.1103/PhysRevE.72.016609 PACS number(s): 42.81.Dp, 42.25.Fx, 42.65.Jx

The self-similar evolution of parabolic pulses in nonlinear dispersive amplifying media has recently attracted much interest [1–3]. The interplay of normal dispersion, self-phase-modulation (SPM), and gain leads to an amplified, linearly chirped pulse with a parabolic temporal profile which is the asymptotic solution of the amplified nonlinear Schrodinger equation (NLSE) [4,5]. The process resembles the well known chirped pulse amplification (CPA) technique, except that the stretching and amplification of the pulse take place simultaneously and automatically for parabolic chirped pulse amplification (PCPA). PCPA has attracted theoretical attention not only because of the existence of an analytical asymptotic solution with elegant self-similarity, but also because it is a useful approach to obtaining high energy and high power pulses from fiber amplifiers [2,3]. Recently, this concept has been extended to fiber lasers [6] and solid state oscillators [7]. Numerical simulations and experiments have demonstrated that stable self-similar parabolic pulse trains can exist in a laser cavity. Due to the linear chirp of the self-similar pulse (also known as a similariton) the output pulse energies can be increased by at least an order of magnitude. It should be noted that PCPA relies on the combination of either normal dispersion and positive nonlinearity (i.e., SPM) or anomalous dispersion and negative nonlinearity.

To date, all the work on parabolic propagation has focused on temporal pulse evolution: the temporal evolution shaped by dispersion and nonlinearity. On the other hand it is well known that diffraction of paraxial optical beams is analogous to dispersive propagation of quasimonochromatic pulses in dielectric media [8]. Both processes are described by equations that are nearly identical in form. Given the space-time duality, a question that naturally arises is this: does there exist a self-similar spatial parabolic *beam*, i.e., a spatial analog to the temporal parabolic pulse? Although diffraction and dispersion are analogous in the two propagation problems, there are differences that make the question interesting. Dispersion is dependent on the waveguide structure as well as the material itself. It can be engineered to provide normal or anomalous dispersion that correspond to positive group velocity dispersion (GVD) or negative GVD, respectively. Since it stretches a temporal pulse, dispersion is a one-dimensional effect. Therefore PCPA is modeled by a (1+1) dimensional amplified NLSE. In contrast, diffraction is primarily a geometrical effect and its fixed sign makes it always equivalent to anomalous dispersion. Since it happens in space, diffraction of a beam can be either one dimensional for the propagation in a planar waveguide, or two dimensional in a bulk medium, in which case the propagation problem corresponds to a (2+1)-dimensional amplified NLSE.

In the following, we first consider solutions to the (1+1)-dimensional problem in a unified formulation making explicit the space-time duality. We then solve the (2+1)-dimensional amplified NLSE demonstrating the existence of self-similar beams in bulk amplifying media.

The propagation equation for a pulse propagating in a fiber amplifier or of a cw paraxial beam in a dielectric planar waveguide amplifier with a Kerr nonlinearity corresponds to a (1+1)-dimensional amplified NLSE:

$$\frac{\partial \varepsilon}{\partial z} = -ia\frac{\partial^2 \varepsilon}{\partial \xi^2} + ik_0 n_2 |\varepsilon|^2 \varepsilon + \frac{g}{2}\varepsilon, \tag{1}$$

where $\varepsilon(z,\xi)$ is the electric field envelope, k_0 the wave number in the vacuum, n_2 the nonlinear index coefficient due to the Kerr nonlinearity, and g is the gain [9]. The variable ξ represents either the lateral transverse dimension in the case of diffraction in a planar waveguide or the local time in the case of pulse propagation. The terms on the right-hand side denote the dispersion or diffraction, SPM, and gain, respectively. The parameter a represents $-1/(2n_0k_0)$ for the waveguide amplifier and $\beta_2/2$ for the fiber amplifier, where n_0 is the refractive index of the waveguide in the spatial problem and β_2 is the GVD in the temporal problem. The parabolic pulse asymptotic solution of Eq. (1) was obtained in [4] via the introduction of self-similarity variables. In order to clarify the physical interactions behind self-similar propagation and to set the stage for our (2+1)-dimensional solution, we employ here a different method motivated by Ref. [10]. We separate ε into a real amplitude A and a phase $\phi, \varepsilon(z,\xi) = A(z,\xi)\exp[i\phi(z,\xi)]$ which transforms Eq. (1) into the following coupled equations in A and ϕ:

$$\frac{\partial\phi}{\partial z} = -\frac{a}{A}\frac{\partial^2 A}{\partial\xi^2} + k_0 n_2 A^2 + a\left(\frac{\partial\phi}{\partial\xi}\right)^2, \tag{2a}$$

$$\frac{\partial A^2}{\partial z} = 2a\frac{\partial}{\partial\xi}\left(A^2\frac{\partial\phi}{\partial\xi}\right) + gA^2. \tag{2b}$$

Three terms on the right-hand side of Eq. (2a) contribute to the evolution of the phase. To compare the contributions from the first two terms, we introduce a parameter $N(z,\xi)$ as

$$N^2(z,\xi) = \left| k_0 n_2 A^2 / \left(\frac{a}{A}\frac{\partial^2 A}{\partial\xi^2}\right)\right| = \frac{k_0 n_2}{a}\left| A^3 / \frac{\partial^2 A}{\partial\xi^2}\right|. \tag{3}$$

It can be shown that given a Gaussian pulse as input, $N^2(0,0)$ is equal to the ratio between the dispersion length and nonlinear length. For a Gaussian beam on the other hand, this quantity is equal to the ratio of the Rayleigh range to the nonlinear length. Therefore, $N(0,0)$ is the soliton order number of the corresponding NLSE [9], and N^2 indicates the relative strength of the nonlinearity to the diffraction or dispersion contributing to the phase advance with propagation.

Under the assumption $N^2 \gg 1$, the first term on the right-hand side of Eq. (2a) can be neglected to give

$$\frac{\partial\omega_c}{\partial z} = -\frac{\partial}{\partial\xi}(k_0 n_2 I + a\omega_c^2), \tag{4a}$$

$$\frac{\partial I}{\partial z} = -2a\frac{\partial}{\partial\xi}(I\omega_c) + gI, \tag{4b}$$

where the intensity function $I = A^2$ and the chirp function $\omega_c = -\partial\phi/\partial\xi$ have been introduced.

Equations (4a) and (4b) have the following self-similar parabolic solution if $an_2 > 0$:

$$I = A_0^2 \exp(2gz/3)[1 - \xi^2/\xi_P^2(z)],$$

for $|\xi| \leq \xi_P(z)$, and $I=0$ for $|\xi| > \xi_P(z)$ where $\xi_P(z) = 6A_0(k_0 n_2 a)^{1/2}\exp(gz/3)/g$ defines the effective pulse or beam width. A_0 is determined by the pulse energy or the beam power at the propagation origin $z=0$. Note that the parabolic solution has a linear chirp, that is, $\omega_c = g\xi/(6a)$, $|\xi| \leq \xi_P(z)$. By substitution of the parabolic solution into Eq. (3), the assumption $N^2 \gg 1$ can be justified *a posteriori* since we have $N^2(z,\xi) = 36A_0^4 k_0 n_2 a \exp(4gz/3)|1 - \xi^2/\xi_P^2|^3/g^2 \gg 1$ given enough propagation distance. The chirp parameter defined as $C(z,\xi) = \partial\omega_c/\partial\xi$ is $C(z,\xi) = g/(6a)$. It is surprising that the chirp parameter is independent of propagation distance. As is well known, pure SPM increases the beam or pulse bandwidth while keeping the beam or pulse width unchanged, thus leading to an increment of the chirp parameter with propagation. On the other hand, diffraction or dispersion only extends the beam or pulse width and keeps the bandwidth constant. Thus the diffraction or dispersion tends to reduce the chirp parameter. Substitution of $\omega_c = g\xi/(6a)$ into Eq. (4a) yields $\partial(k_0 n_2 I + a\omega_c^2)/\partial\xi = 0$. This clearly shows that during the propagation, the increment and reduction of the chirp parameter cancel each other to reach a local dynamic balance. The role of the amplification is to elevate the intensity of the beam or pulse and to ensure that $k_0 n_2 I$ dominates the $a/A\partial^2 A/\partial\xi^2$ contribution to the phase accumulation with propagation. Evidently, it is the interaction of the dispersion or diffraction, nonlinearity, and gain that sustains the self-similar parabolic evolution and keeps the chirp parameter constant. For the case without gain, a $(1+1)$-dimensional optical soliton can be achieved due to the dynamic balance of the nonlinearity and dispersion or diffraction governed by NLSE, where the constant is the shape of the soliton and its width.

The above analysis demonstrates the existence of the parabolic beam in a planar waveguide amplifier with a negative nonlinearity. Indeed, the result is not surprising due to its equivalence to a pulse fiber amplifier, both of which are characterized by a $(1+1)$-dimensional amplified NLSE. The duality predicts the parabolic beam even before we wrote down the equation. However, intuition alone cannot tell us whether such a parabolic beam exists in a bulk medium, where it would be described by a $(2+1)$-dimensional amplified NLSE. For example, stable soliton solutions to the $(2+1)$-dimensional NLSE do not exist if the nonlinearity does not saturate; thus it is not obvious that the existence of self-similar solutions in $2+1$ dimensions can be inferred by analogy.

We are interested in the solutions for an initial Gaussian beam $\varepsilon(0,r) = [2P_{in}/(\pi w_0^2)]^{1/2}\exp(-r^2/w_0^2)$ where P_{in} is the beam input power, w_0 the spot size. The corresponding Rayleigh range is given as $z_R = \pi w_0^2 n_0/\lambda_0$ where λ_0 is the wavelength in the vacuum and n_0 the refractive index of the medium. Under the following normalization: $z \to 4z_R z$, $r \to w_0 r$, $g \to g/(4z_R)$, $\varepsilon = U\sqrt{2P_{in}/(\pi w_0^2)}$, the dimensionless form of such an equation in cylindrical coordinates can be written as [11]

$$i\frac{\partial U}{\partial z} = -\left(\frac{1}{r}\frac{\partial U}{r} + \frac{\partial^2 U}{\partial r^2}\right) - \gamma|U|^2 U + i\frac{g}{2}U, \tag{5}$$

where $\gamma \equiv 8P_{in}/P_{cr}$ and the critical power for beam collapse is $P_{cr} = 0.159\lambda_0^2/(n_0 n_2)$. Following the same approach to the $(1+1)$-dimensional amplified NLSE, we can define an N^2 parameter for the $(2+1)$-dimensional equation as $N^2 \equiv |\gamma|U|^2 U/[1/r\,\partial/\partial r(r\partial U/\partial r)]|$. Under the assumption $N^2 \gg 1$, we obtain a parabolic solution to Eq. (5):

$$U(z,r) = (g/2)^{1/2}/(2|\gamma|^{1/4})\exp(gz/4)[1 - r^2/r_p^2(z)]^{1/2}\exp[i\varphi(z,r)],$$

$$0 \leq r \leq r_p(z), \tag{6}$$

and $U(z,r)=0$ for $r > r_p(z)$ where $r_p(z) = 2\exp(gz/4)|\gamma|^{1/4}/(g/2)^{1/2}$. This corresponds to a beam diffracted in two transverse dimensions with a parabolic intensity profile and a quadratic phase given by

$$\varphi(z,r) = \varphi_0 + |\gamma|^{1/2}\exp(gz/2)/4 + gr^2/16, \tag{7}$$

where φ_0 is an arbitrary constant. The local spatial frequency is given by

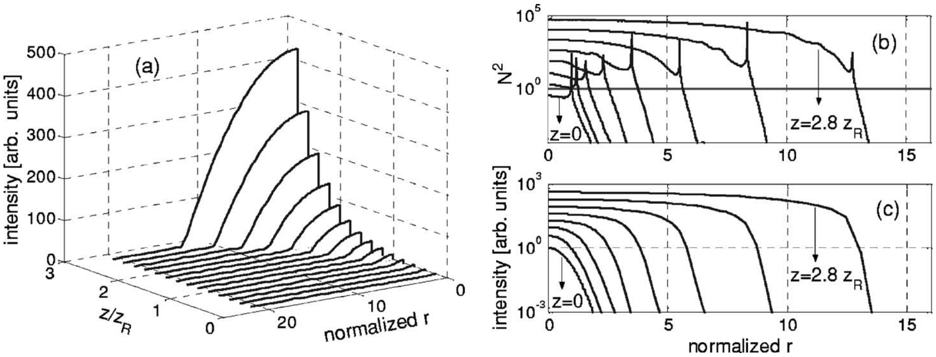

FIG. 1. (a) evolution of a Gaussian input beam into a parabolic beam in a bulk amplifier with negative nonlinearity. (b), (c) and intensity on logarithmic scale in 0.4- increments.

$$\rho_s = -\frac{\partial\varphi(z,r)}{\partial r} = -gr/8.$$

The spatial frequency spectrum of this parabolic beam is obtained with the stationary phase method [12], yielding another parabolic function of Ω, the radius in the spatial frequency domain:

$$\Phi(z,\Omega) = 2\sqrt{2}|\gamma|^{1/4}/\sqrt{g}\exp(gz/4)[1-\Omega^2/\Omega_p^2(z)]^{1/2} \times\exp[i\psi(z,\Omega)],$$

$$0 \leqslant \Omega \leqslant \Omega_p(z), \quad (8)$$

and $\Phi(z,\Omega)=0$ for $\Omega > \Omega_p(z)$ where $\Omega_p(z) = (g/2)^{1/2}|\gamma|^{1/4}\exp(gz/4)/4$ and the spatial frequency spectrum also has a quadratic phase as

$$\psi(z,\Omega) = \varphi_0 + \pi/2 + |\gamma|^{1/2}\exp(gz/2)/4 - 4\Omega^2/g. \quad (9)$$

By analogy to the group delay of a pulse defined as the derivative of the spectral phase with respect to frequency, an equivalent spatial group separation is given as

$$D_s = \frac{\partial\psi(z,\Omega)}{\partial\Omega} = -8\Omega/g.$$

We have confirmed these analytical results by numerical simulations of Eq. (5) based on the quasi-fast Hankel transform [13]. In the simulation, an initially Gaussian beam is incident on a bulk medium amplifier with a gain coefficient of 18 dB/z_R and $\gamma=-1.5$. The negative γ indicates the negative nonlinearity. Figure 1(a) shows the evolution of the beam intensity profile from the numerical simulation. It clearly illustrates that the input Gaussian beam stretches with amplification and gradually evolves into a parabolic beam. The evolution enters the parabolic regime at a propagation distance of 1.8-z_R. In order to confirm our assumption that the parabolic beam is achieved when $N^2 \gg 1$ is satisfied, N^2 and the corresponding beam intensity profile on logarithmic scale in 0.4-Z_R increments are shown in Figs. 1(b) and 1(c). We can see that N^2 increases and extends dramatically along the propagation. In the region of $N^2 \gg 1$, the beam profile has a parabolic shape [the relatively flat part of the curve in the Fig. 1(c), since the y axis is plotted on a logarithmic scale]. In the region where $N^2 \gg 1$ is not satisfied, our assumption fails and the beam profile curve shows a fast decreasing wing with much lower intensity. Both analytical results (solid curve) and the numerical simulations (open circles) after 2.9-z_R of propagation are shown in Fig. 2. It can be seen from Figs. 2(b) and 2(d) that the initial input Gaussian beam has asymptotically evolved into a parabolic beam with a parabolic spatial frequency spectrum. In Figs. 2(a) and 2(c), the linear dependence of ρ_s on r and D_s on Ω indicates the quadratic phase for both the beam and its spectrum. The deviations between the analytical results and the simulations at the low power wings are expected where the $N^2 \gg 1$ does not hold. The excellent agreement between two methods verifies our asymptotic parabolic beam solutions.

In fact, the existence of a gain is not the only way to realize the assumption $N^2 \gg 1$. From its definition, such a condition can be achieved if the input pulse/beam has a strong enough intensity. Reference [10] shows that in the high intensity limit, the (1+1)-dimensional NLSE also supports self-similar propagation of a parabolic pulse in a normally dispersive fiber without gain. Those theoretical predictions were confirmed experimentally by further propagation of the parabolic pulse, which is obtained from a fiber amplifier, through a piece of passive fiber [1]. We expect that the same phenomenon happens for the (2+1)-dimensional

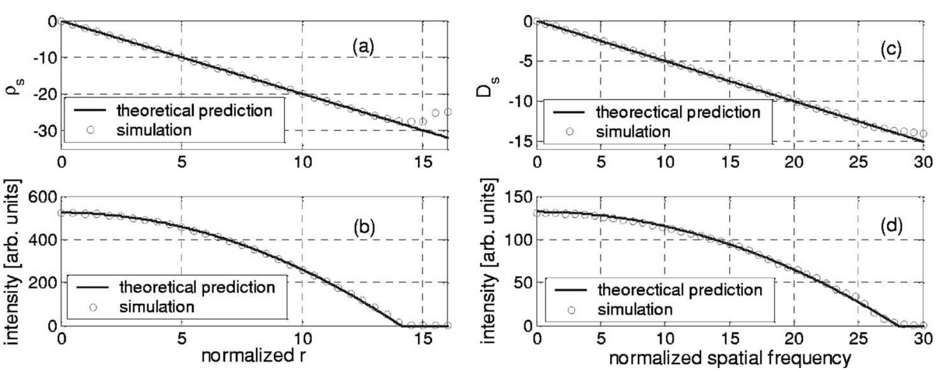

FIG. 2. Comparison between the numerical simulation and the asymptotic parabolic beam results. (a) local spatial frequency, (b) beam intensity profile, (c) spatial group separation, and (d) spatial frequency spectrum

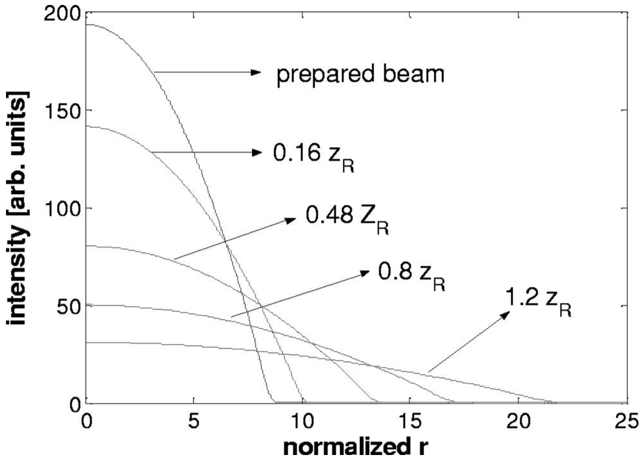

FIG. 3. Self-similar propagation of a high intensity parabolic beam without gain.

NLSE; Fig. 3 shows the simulation results. At first, the strong parabolic beam at a propagation distance of 2.4-z_f is selected from Fig. 1(a). Taking this parabolic beam as the input, we run the simulation with the gain turned off, keeping other parameters unchanged. Without gain, the beam intensity quickly drops with the lateral spread of the beam during propagation, while the parabolic shape of the beam is maintained. Certainly the continuous reduction of the beam intensity will eventually lead to N^2 comparable to one and therefore the parabolic propagation will cease.

It should be noted that the parabolic beam generation and propagation rely on a sign match between diffraction and Kerr nonlinearity. Since diffraction is equivalent to anomalous dispersion, a negative nonlinearity is required. Fortunately, negative nonlinearities have been investigated experimentally for more than a decade, with most studies focused on spatial dark solitons and optical switching [14–17]. Potential materials or systems include semiconductors (such as ZnSe) [14], sodium vapor [15], electromagnetically induced transparency (EIT) material [16], and polymer [17]. An alternative to the real negative nonlinearity is an effective negative nonlinearity produced by cascaded quadratic processes [18]. Properly combined with gain, the generation of a parabolic beam from such media may be anticipated. Recently, it has been demonstrated that diffraction in a waveguide array can be reversed [19]; this opens another possible approach to generating and propagating a parabolic beam utilizing the reversed diffraction and a positive nonlinearity.

In conclusion, we have solved the $(1+1)$-dimensional and $(2+1)$-dimensional amplified NLSE's using a method which illuminates the physical origin of self-similar propagation. We have verified by numerical simulations that our asymptotic solutions predict the formation and propagation of self-similar parabolic beams. The specific roles of diffraction, SPM, and gain during the propagation of the parabolic beam have been identified. It has also been shown that in the high-intensity limit, a parabolic beam can self-similarly propagate a certain distance even without a gain. Possible experimental implementations have been discussed. We note that the beam propagation solutions found here turn out to be directly analogous to higher dimensional asymptotic parabolic solutions which have been found for the amplified Gross-Pitaevskii (GP) equation describing the growth of a Bose-Einstein condensate [20]. As a special class of amplified GP equation without the potential term, a $(3+1)$-dimensional amplified NLSE equation can be used to describe the propagation of a pulsed beam through a bulk amplifier with anomalous dispersion and negative Kerr nonlinearity. This implies that a parabolic-pulsed parabolic beam can be obtained. Parabolic beams may prove to be useful as well as interesting: for example, a parabolic profile falls much faster from the peak to the wings compared to a Gaussian or a hyperbolic secant, which could be useful for applications like laser machining. By analogy to the similariton for a pulsed oscillator [7], it might be possible to build high power lasers with the help of the parabolic chirped beam amplification technique.

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